

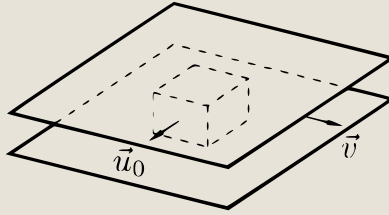
Squeezed brick

Igor Kravchenko

November 2025

Written for [Physics Cup – TalTech 2026 \(Problem 1\)](#).

A small brick is squeezed between two parallel planes in zero-gravity. The planes are perpendicular to the axis z . The lower plane always stays in rest, whilst the upper one is moving with the constant velocity $\vec{v}$ along the axis x , see the figure. Initially the velocity of the brick $\vec{u}_0$ is perpendicular to $\vec{v}$ and lies in axis y . The friction between the planes and the brick is dry with the same coefficient of kinetic friction. Given the initial acceleration of the brick a_0 , find the displacement l_y of the brick along the axis y after a long time. Obtain an expression of l_y for arbitrary speeds v and u_0 , and evaluate the answer for $v = u_0 = 1$ m/s, and $a_0 = 1$ m/s² with four digits.



Partial solution

Answer:

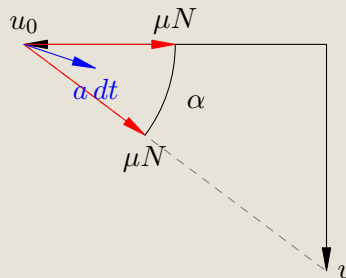
$$l_y = \frac{u_0^2 A^3 B}{v} \int_0^{\pi/2} \frac{A + v \sin \varphi + u_0}{(A + v \sin \varphi)^3} d\varphi,$$

where $A = \sqrt{u_0^2 + v^2} - u_0$, $B = \frac{2}{a_0} \cos \left(\frac{1}{2} \arctan \left(\frac{v}{u_0} \right) \right)$.

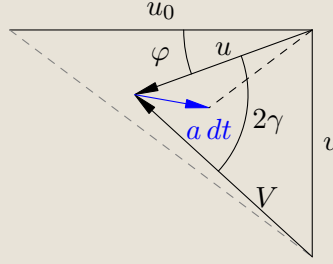
Evaluating: $l_y \approx 0,6631$ m.

Vector diagram

At the start of the problem we have the following vector diagram.



The acceleration is the angle bisector of the relative velocity vectors. Then at any time we have the following diagram (qualitative).



The elementary displacement along y -coordinate:

$$dl_y = u dt \cos \varphi. \quad (1)$$

The elementary increasing of the angle φ :

$$d\varphi = \frac{a dt \sin \gamma}{u}. \quad (2)$$

The acceleration:

$$a = \frac{2\mu N \cos \gamma}{m}. \quad (3)$$

where $\frac{\mu N}{m} = \frac{a_0}{2 \cos \left(\frac{1}{2} \arctan (v/u_0) \right)}$.

The cosine theorem for the uvV -triangle:

$$V^2 = u^2 + v^2 - 2uv \cos(\pi/2 - \varphi). \quad (4)$$

The changungs of the relative velocity vectors are equal, since the acceleration is the angle bisector of them, then

$$V - \sqrt{u_0^2 + v^2} = u - u_0. \quad (5)$$

The sine theorem for the uvV -triangle:

$$\frac{V}{\sin(\pi/2 - \varphi)} = \frac{v}{\sin 2\gamma}. \quad (6)$$

The system of equations (1)–(6) can be solved.

$$dl_y = \frac{u_0^2 A^3 B}{v} \cdot \frac{A + v \sin \varphi + u_0}{(A + v \sin \varphi)^3} d\varphi,$$

where $A = \sqrt{u_0^2 + v^2} - u_0$, $B = \frac{2}{a_0} \cos \left(\frac{1}{2} \arctan \left(\frac{v}{u_0} \right) \right)$.

Integral

It is clear, that for the final l_y the angle φ is changed from 0 to $\pi/2$, then:

$$l_y = \frac{u_0^2 A^3 B}{v} \int_0^{\pi/2} \frac{A + v \sin \varphi + u_0}{(A + v \sin \varphi)^3} d\varphi.$$

For the numeric data:

$$l_y \approx 0,6631 \text{ m.}$$

Complete solution

Answer:

$$l_y = \frac{m}{2\mu N} \frac{b^2}{\varepsilon \sqrt{\varepsilon^2 - 1}} \frac{1}{2} \left(\varepsilon^2 \frac{u_0}{b} \sqrt{u_0^2/b^2 + 1} + (\varepsilon^2 - 2) \sinh^{-1}(u_0/b) \right)$$

where

$$\begin{aligned} \frac{m}{2\mu N} &= \frac{\cos \left(\frac{1}{2} \arctan(v/u_0) \right)}{a_0}, \\ b^2 &= \frac{u_0}{2} \left(\sqrt{v^2 + u_0^2} - u_0 \right), \\ \varepsilon &= \frac{v}{\sqrt{v^2 + u_0^2} - u_0}. \end{aligned}$$

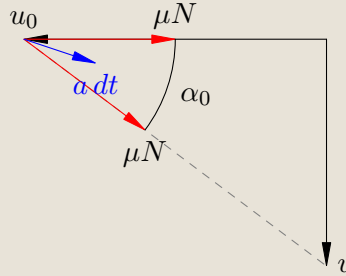
For the **numeric** data ([wolfram alpha](#)):

$$l_y \approx 0,6631 \text{ m.}$$

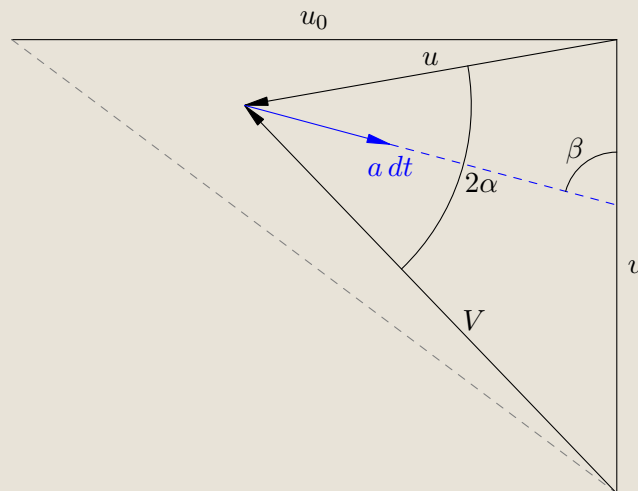
This numeric answer is identical to that from the partial solution.

Vector diagram

At the start of the problem we have the following vector diagram.



The acceleration is the angle bisector of the relative velocity vectors. Then at any time we have the following diagram (qualitative).



The elementary displacement dl_y along y -coordinate:

$$dl_y = u_y dt = u_y \frac{du_y}{a_y} = -u_y \frac{du_y}{a \sin \beta} = -u_y \frac{du_y}{\frac{2\mu N}{m} \cos \alpha \sin \beta}, \quad (1)$$

where u_y and a_y are the projections of the velocity and the acceleration onto the y -axis respectively, dt is a small time increment, du_y is a small increment of the projection u_y , a — the acceleration, β is the angle between $\vec{a}$ and $\vec{v}$ (or x -axis), 2α is the angle between $\vec{u}$ and $\vec{V}$ (the relative velocity in the frame of the upper plane), μN — the friction force from every plane, m — the mass.

We have also

$$\frac{2\mu N}{m} = \frac{a_0}{\cos(\alpha_0/2)} = \frac{a_0}{\cos\left(\frac{1}{2} \arctan(v/u_0)\right)} \quad (2)$$

from the 2nd Newton law for the initial conditions.

Now, before summing we have find the relationship between α , β and u_y .

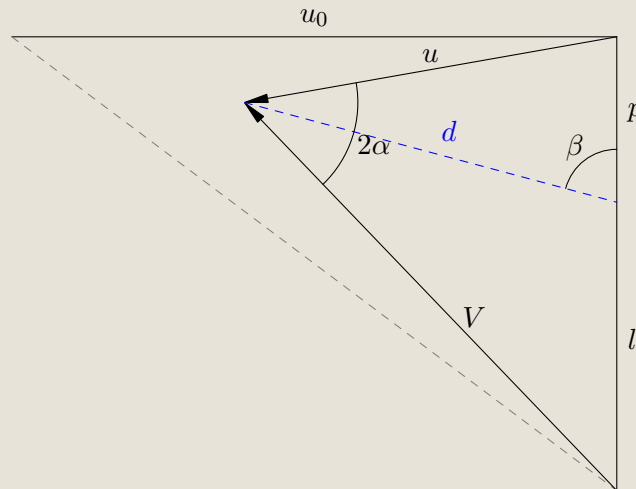
Hyperbola

The changings of the relative velocity u and V are equal, since the acceleration is the angle bisector of them, so the endpoint of the velocity vector $\vec{u}$ has hyperbolic path on the velocity diagram above.

It is clear, that the foci of this hyperbola are lying at the start- and the endpoint of the vector $\vec{v}$.

The angle β is the angle between the tangent line to the hyperbola and the its major axis. The angle α is half of the angular distance between the two foci of the hyperbola.

First, we are going to find the relationship between α and β . Let us consider four applications of the law of cosines for two suitably chosen triangles.



For the triangles "udp" and "Vdl" we have:

$$l^2 = V^2 + d^2 - 2Vd \cos \alpha, \quad (3)$$

$$V^2 = l^2 + d^2 - 2ld \cos(\pi - \beta), \quad (4)$$

$$p^2 = u^2 + d^2 - 2ud \cos \alpha, \quad (5)$$

$$u^2 = p^2 + d^2 - 2pd \cos \beta. \quad (6)$$

Solution of the system of equations gives

$$\boxed{\varepsilon \cos \beta = \cos \alpha}, \quad (7)$$

where $\varepsilon = c/a$ is eccentricity of the hyperbola. (We denote $2c = l + p$ as the distance between the foci, $2a = V - u$ as the absolute difference of the distances to the foci from an arbitrary point on the hyperbola.)

We know two more formulas for hyperbola — the equation of a hyperbola and the equation of a tangent line to the hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad (8)$$

$$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1. \quad (9)$$

where $b^2 = c^2 - a^2$. For our diagram we assume also the following

$$2a = \sqrt{v^2 + u_0^2} - u_0, \quad (10)$$

$$2c = v. \quad (11)$$

From (8) and (9) we can obtain

$$\tan^2 \beta = \frac{(b^2 + y^2)(\varepsilon^2 - 1)}{y^2}, \quad (12)$$

what for our velocity diagramm we can rewrite as the following

$$\boxed{\tan^2 \beta = \frac{(b^2 + u_y^2)(\varepsilon^2 - 1)}{u_y^2}} \quad (13)$$

(replacing $y \rightarrow u_y$).

Summing

With some additional trigonometric formulas we obtain from (7) and (13)

$$\cos \alpha \sin \beta = \frac{\varepsilon u_y \sqrt{(b^2 + u_y^2)(\varepsilon^2 - 1)}}{u_y^2 + (b^2 + u_y^2)(\varepsilon^2 - 1)} \quad (14)$$

for the (1).

Now we rewrite (1)

$$dl_y = -\frac{m}{2\mu N} \frac{1}{\varepsilon \sqrt{\varepsilon^2 - 1}} \left(\frac{u_y^2 + (b^2 + u_y^2)(\varepsilon^2 - 1)}{\sqrt{(b^2 + u_y^2)}} \right) du_y \quad (15)$$

Now we make nondimensionalysing by using the substitution $z \equiv u_y/b$

$$dl_y = -\frac{m}{2\mu N} \frac{b^2}{\varepsilon \sqrt{\varepsilon^2 - 1}} \left(\frac{z^2 + (1 + z^2)(\varepsilon^2 - 1)}{\sqrt{(1 + z^2)}} \right) dz \quad (16)$$

It is clear, that for the final l_y the projection u_y is changed from u_0 to 0, therefore z is changed from u_0/b to 0. Then

$$l_y = -\frac{m}{2\mu N} \frac{b^2}{\varepsilon \sqrt{\varepsilon^2 - 1}} \int_{u_0/b}^0 \left(\frac{z^2 + (1 + z^2)(\varepsilon^2 - 1)}{\sqrt{(1 + z^2)}} \right) dz \quad (17)$$

After finding [indefinite](#) and definite (manually) integral assuming ε and z are real:

$$l_y = \frac{m}{2\mu N} \frac{b^2}{\varepsilon \sqrt{\varepsilon^2 - 1}} \frac{1}{2} \left(\varepsilon^2 \frac{u_0}{b} \sqrt{u_0^2/b^2 + 1} + (\varepsilon^2 - 2) \sinh^{-1}(u_0/b) \right) \quad (18)$$

where

$$\frac{m}{2\mu N} = \frac{\cos \left(\frac{1}{2} \arctan(v/u_0) \right)}{a_0}, \quad (19)$$

$$b^2 = \frac{u_0}{2} \left(\sqrt{v^2 + u_0^2} - u_0 \right), \quad (20)$$

$$\varepsilon = \frac{v}{\sqrt{v^2 + u_0^2} - u_0}. \quad (21)$$

For the numeric data ([wolfram alpha](#)):

$$l_y \approx 0,6631 \text{ m.}$$